Paths of FRW brane models

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Dynamics of brane-world models of dark energy is reviewed. We demonstrate that simple dark energy brane models can be represented as 2-dimensional dynamical systems of a Newtonian type. Hence a fictitious particle moving in a potential well characterizes the model. We investigate the dynamics of the brane models using methods of dynamical systems. The simple brane-world models can be successfully unified within a single scheme – an ensemble of brane dark energy models. We characterize generic models of this ensemble as well as exceptional ones using the notion of structural stability (instability). Then due to the Peixoto theorem we can characterize the class of generic brane models. We show that global dynamics of the generic brane models of dark energy is topologically equivalent to the concordance ΛCDM model. We also demonstrate that the bouncing models or models in which acceleration of the universe is only transient phenomenon are non-generic (or exceptional cases) in the ensemble. We argue that the adequate brane model of dark energy should be a generic case in the ensemble of FRW dynamical systems on the plane.

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I. INTRODUCTION

Recent observations of distant supernovae type Ia (SNIa) [1, 2] as well as other current observations of cosmic microwave background radiation (CMBR) anisotropies indicate that our universe is almost flat and stay in the accelerated phase of expansion [3, 4]. Principally there are two alternatives of explanation properties of current universe. In the first approach the universe is accelerating due to presence of mysterious energy called dark energy X, of unknown origin, which violates the strong energy condition $\rho_X + 3p_X > 0$, where ρ_X and p_X are energy density and pressure of dark energy, respectively. Alternative to the dark energy idea is the explanation of acceleration of the universe based on some modification to the Friedmann-Robertson-Walker (FRW) dynamics arising from new physics (i.e., extra dimensions). In this approach dark energy is the manifestation of modified gravitational dynamics of the 4-dimensional brane leading to a self-accelerated universe. In this conception which is called brane dark energy new physics mimic on the phenomenological level dark energy or effects of extra dimensions manifest themselves as a modification to the Friedmann equation which govern the evolution of the 4-dimensional brane localized in the higher-dimensional bulk space. In contrast to the classical Kaluza-Klein theories (when extra dimensions, which manifest themselves at high energies, are either compact and have finite volume), theories with infinite volume of extra dimensions modify the late time cosmological evolution of the universe and gravitational dynamics is preserved at short distances.

The hierarchy problem [5, 6, 7] and the problem of present acceleration of the Universe [8, 9] also motivate all theories with large extra dimensions. In the literature of the subject there are many different propositions of dark energy of brane origin (for review see [10]). Different brane-world scenarios were tested by recent astronomical measurements of distant supernovae Ia [11] as well as other observations from Type IIb radio galaxies and X-ray gas mass fraction [12, 13]. The stringent constraint on the model parameters can be also obtained from baryon oscillation peak measurements [14, 15, 16].

The main aim of this paper is to investigate dynamics of the most popular brane dark energy models using methods of dynamical systems. We investigate a class of the simplest brane models in which our observable universe is a $(3+1)$-dimensional brane and a 3-space is homogeneous and isotropic. The physical space is embedded in a $(4+1)$-dimensional space called a bulk space. For all these models we obtain an analog of the classical Friedmann equation

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with some additional terms which are functions of both the scale factor a and energy density of matter on the brane (see Table I). If we assume that the energy density satisfies the conservation condition $\dot{\rho} = -3H(\rho+p)$, where $p = p(a)$ is the equation of state, then $\rho = \rho(a)$ and the modified FRW equation on the brane can be represented by

$$H^2 + \frac{k}{a^2} \equiv \frac{\rho_{\text{eff}}}{3}, \tag{1}$$

where $H = (\ln a)^{\cdot}$ is Hubble's function, k is the curvature index, $\rho_{\text{eff}} = \rho_{\text{eff}}(a)$ is the effective energy density. For the concordance ΛCDM model $\rho_{\text{eff}} = \rho_{\text{m},0}(a/a_0)^{-3} + \Lambda$, $p_{\text{eff}} = 0 - \Lambda$. Without lost of degree of generality one can choose $a_0 = 1$ – as the present value of the scale factor; $1 + z = a_0/a$, where z is redshift.

Equation (1) can be rewritten to the new form analogous to the first integral of the energy for a unit mass particle moving in the potential well

$$\frac{\dot{a}^2}{2} + V(a) \equiv 0, \tag{2}$$

where a dot denotes the differentiation with respect to the rescaled time $t \to \tau\colon |H_0|\mathrm{d}t = \mathrm{d}\tau$ and $V(a) = -\frac{1}{2}\{\Omega_{\text{m},0}a^{-1} + \Omega_{k,0} + \Omega_{\Lambda,0}a^2\}$; $\Omega_{\text{m},0} = \frac{\rho_{\text{m},0}}{3H_0^2}$, $\Omega_{\Lambda,0} = \frac{\Lambda}{3H_0^2}$ are density parameters for matter and the cosmological constant on the brane, $V(a) = -\frac{\rho_{\text{eff}}a^2}{6}$.

Equation (2) reduces the problem of dynamics of a cosmological model to the standard problem of classical mechanics – motion of a particle of a unit mass in the potential well $V(a)$ on the distinguished "zero energy level".

For the FRW brane cosmological models there is a counterpart of the Friedmann first integral (2) in the form

$$\frac{\dot{a}^2}{2} + V(a) + V_{\text{brane}}(a) = 0, \tag{3}$$

where $V_{\text{brane}}(a) = -\frac{\rho_{\text{brane}}a^2}{6}$, $V(a) = -\frac{\rho_{\text{eff}}a^2}{6}$.

Both relations (2) and (3) play the role of a first integral of motion for the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\tfrac{\partial V}{\partial x}, \end{aligned} \tag{4}$$

which has the form of a 2-dimensional dynamical system of a Newtonian type and $x = a/a_0$.

The classical motion of the system is restricted to the domain admissible for motion:

$$\mathcal{D}_0 = \{x\colon V \leq 0\}. \tag{5}$$

The evolution of the system has been visualized in 2-dimensional phase space $(x, \dot{x})$. There are two types of solutions of (4): 1) a singular one which corresponds to the critical points of system (4) such that $y_0 = 0$, and $(\partial V/\partial x)_{x_0} = 0$; 2) a nonsingular one which lies on the algebraic curves determined by the Friedmann first integral $y^2/2 + V(x) = 0$. It is convenient to extract the curvature term from the potential function. Then we can regard evolution of the system on a constant energy level $E = \frac{1}{2}\Omega_{k,0}$.

An advantage of dynamical system methods is a investigation of all admissible solutions for all initial conditions. A picture of all evolutional paths represented by phase curves and critical points form phase portrait – global visualization of dynamics. The trajectory of the flat model $\Omega_{k,0} = 0$ divides phase portrait in to two subsets occupied by open models ($\Omega_{k,0} > 0$) and by closed models ($\Omega_{k,0} < 0$).

The character (type) of the critical points is, following the Hartman-Grobman theorem, determined from the linearized system

$$\begin{aligned} (x - x_0)^{\cdot} &= y, \\ (y - 0)^{\cdot} &= -(\tfrac{\partial V}{\partial x})_{x=x_0}(x - x_0), \end{aligned} \tag{6}$$

around the critical point (which is a static critical point in any case in a finite domain of the phase space). Therefore the characteristic equation of the linearization matrix is

$$\lambda^2 + \det A = 0, \tag{7}$$

where $\operatorname{tr} A = 0$, $\det A = V_{xx}(x_0)$.

From equation (7) we obtain that only two types of critical points are admissible for the system of a Newtonian type, namely saddles if $V_{xx}(x_0) < 0$ or centers in opposite case of $V_{xx}(x_0) > 0$. The case of $V_{xx}(x_0) = 0$ (a inflection

TABLE I: Classification of simple dark energy brane models in terms of $H(x)$ relation and $V_{\mathrm{brane}} = -\frac{\rho_{\mathrm{brane}} a^2}{6} = -\frac{1}{2}\Omega_{\mathrm{brane}}(a)a^2$.

case	model	$(\frac{H}{H_0})^2 = f(x)$	V_{brane}
I	Dvali-Turner model	$(1-\Omega_{\mathrm{m},0})(\frac{H}{H_0})^{\alpha} + \Omega_{\mathrm{m},0}x^{-3}$	
II	Dvali-Gabadadze-Porrati model $p = \omega\rho$	$\Omega_{k,0}x^{-2} + \left(\sqrt{\Omega_{rc}} + \sqrt{\Omega_{rc} + \Omega_{\mathrm{m},0}x^{-3(1+\omega)}}\right)^2$	$-\frac{1}{2}\left(\sqrt{\Omega_{rc}} + \sqrt{\Omega_{rc} + \Omega_{m,0}x^{-3(1+\omega)}}\right)^2 x^2$
III	Shtanov-Sahni model $p = \omega\rho$; $\epsilon = \pm 1$	$\Omega_{\Lambda,0} + \Omega_{\mathrm{m},0}x^{-3} + \Omega_{d,r}x^{-4} + \epsilon\|\Omega_{\lambda,0}\|x^{-6(1+\omega)}$	$-\frac{1}{2}\left(\Omega_{\Lambda,0} + \Omega_{m,0}x^{-3} + \Omega_{d,r}x^{-4} + \epsilon\|\Omega_{\lambda,0}\|x^{-6(1+\omega)}\right)x^2$
IV	Sahni-Shtanov Brane I (+) and Brane II (−) models	$\Omega_{\mathrm{m},0}x^{-3} + \Omega_{\sigma} + 2\Omega_l \pm 2\sqrt{\Omega_l}\sqrt{\Omega_{\mathrm{m},0}x^{-3} + \Omega_{\sigma} + \Omega_l + \Omega_{\Lambda_b}}$	$-\frac{1}{2}\left(\Omega_{m,0}x^{-3} + \Omega_{\sigma} + 2\Omega_l \pm 2\sqrt{\Omega_l}\sqrt{\Omega_{\mathrm{m},0}x^{-3} + \Omega_{\sigma} + \Omega_l + \Omega_{\Lambda_b}}\right)x^2$

point) is a degenerated case. In the first case eigenvalues of the linearization matrix are real of opposite sings. In the case of centers (non-hyperbolic critical points) eigenvalues are purely imaginary.

For the conservative system it is useful to develop methods of qualitative investigations of differential equations [17]. The main aim of this approach is to construct phase portraits of the system which contain global information about the dynamics. The phase space (x, y) offers a possibility of natural geometrization of the dynamical behavior. It is in a simple 2-dimensional case structuralized by critical points or non-point closed trajectories (limit cycles) and trajectories joining them. Two phase portraits are equivalent modulo homeomorphism preserving orientation of the phase curves (or phase trajectories). From the physical point of view critical points (and limit cycles) represent asymptotic states (equilibria) of the system. Equivalently, one can look at the phase flow as a vector field

$$\boldsymbol{f} = \left[y, -\frac{\partial V}{\partial x}\right]^T, \tag{8}$$

whose integral curves are the phase curves.

Thanks to Andronov and Pontryagin [18], the important idea of structural stability was introduced into the ensemble [19] of all dynamical systems. A vector field, say $\boldsymbol{f}$, is a structurally stable vector field if there is an $\varepsilon > 0$ such that for all other vector fields $\boldsymbol{g}$, which are close to $\boldsymbol{f}$ (in some metric sense) $\|\boldsymbol{f} - \boldsymbol{g}\| < \varepsilon$, $\boldsymbol{f}$ and $\boldsymbol{g}$ are topologically equivalent. The notion of structural stability is mathematical formalization of intuition that physically realistic models in applications should posse some kind of stability, therefore small changes of the r.h.s. of the system (i.e. vector field) doesn't disturb the phase portrait. For example motion of pendulum is structurally unstable because small changes of vector field (constructed from r.h.s. of the system) of a friction type $\boldsymbol{g} = [y, -\partial V/\partial x + ky]^T$ dramatically changes the structure of phase curves. While for the pendulum, the phase curves are closed trajectories around the center, in the case of pendulum with friction (constant) they are open spirals converging at the equilibrium after infinite time. We claim that the pendulum without friction is structurally unstable. Many dynamicists believe that realistic models of physical processes should be typical (generic) because we always try to convey the features of typical garden variety of the dynamical system. The exceptional (non-generic) cases are treated in principle as less important because they interrupt discussion and do not arise very often in applications [20].

In the 2-dimensional case, the famous Peixoto theorem gives the characterization of the structurally stable vector field on a compact two dimensional manifold [21]. They are generic and form open and dense subsets in the ensemble of all dynamical systems on the plane. If a vector field f is not structurally stable it belongs to the bifurcation set.

The space of all conservative dark energy models can be equipped with the structure of the Banach space with the C^1 metric.

Let V_1 and V_2 be two dark energy models. Then C^1 distance between them in the ensemble is

$$d(V_1, V_2) = \max\left\{\sup_{x\in E} |V_{1,x} - V_{2,x}|, \sup_{x\in E} |V_{1,xx} - V_{2,xx}|\right\}, \tag{9}$$

where E is a closed subset of configuration space. Of course, ensemble of all dynamical systems of Newtonian type on the plane is infinite dimensional functional space and the introduced metric is a so-called Sobolev metric.

While there is no counterpart of the Peixoto theorem in higher dimensions it can be easy to test whether planar polynomial systems, like in considered case, have structurally stable phase portraits. For this aim the analysis of behavior of trajectories at infinity should be performed. One can simply do that using the tools of Poincarè S^2 construction, namely by projection trajectories from the center of the unit sphere $S^2 = \left\{(X, Y, Z) \in \mathbf{R}^3 : X^2 + Y^2 + Z^2 = 1\right\}$ onto the (x, y) plane tangent to S^2 at either the north or south pole.

The vector field f is structurally unstable if:

1. there are non-hyperbolic critical points on the phase portrait,

2. there is a trajectory connecting saddles on the equator of S^2.

In opposite cases if additionally the number of critical point and limit cycles is finite, f is structurally stable on S^2.

Let us consider 2-dimensional dynamical system of a Newtonian type (5). There are three cases of behavior of the system admissible in the neighborhood of the critical point $(x_0, 0)$: $-\partial V/\partial x|_{x_0} = 0$:

- If $(x_0, 0)$ is a strict local maximum of $V(x)$, it is a saddle point;

- If $(x_0, 0)$ is a strict local minimum of $V(x)$, it is a center;

- If $(x_0, 0)$ is a horizontal inflection point of $V(x)$, it is a cusp.

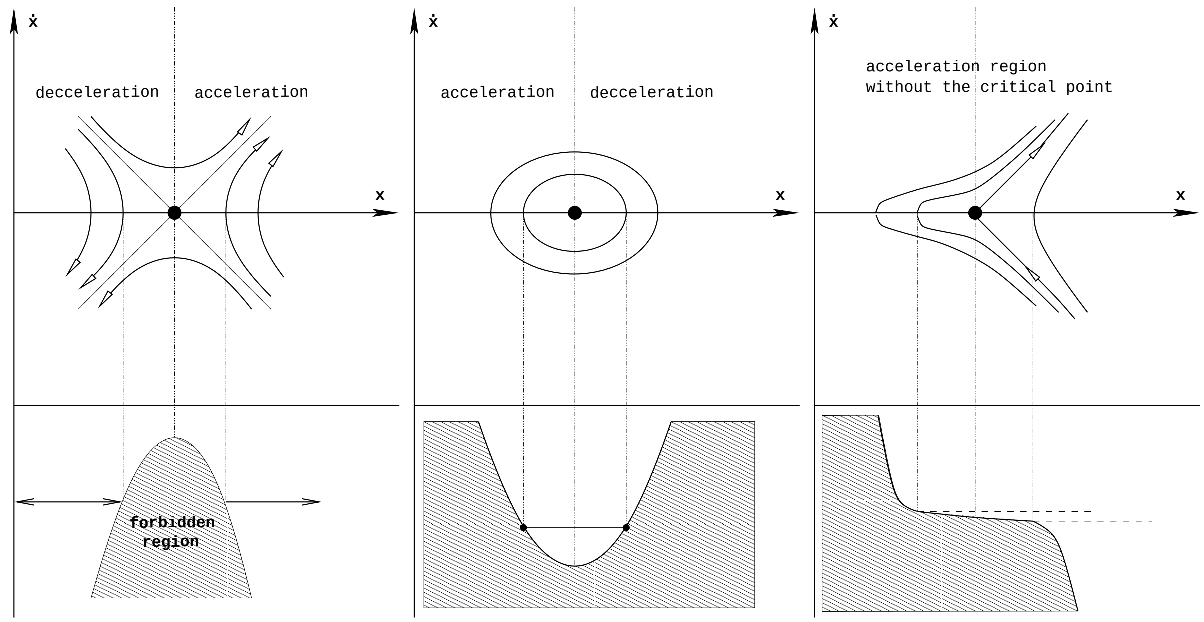


FIG. 1: Three possible types of behavior in the neighborhood of a critical point (from left to right): a saddle, a center and a cusp.

All these cases are illustrated in Fig. 1.

It is a simple consequence of the fact that characteristic equation for linearization matrix at the critical point ($\mathrm{tr}\, A = 0$) is $\lambda^2 + \det A = 0$, where $\det A = \partial^2 V/\partial x^2|_{x_0}$. Therefore the eigenvalues are real of opposite sign for saddle point, and for centers – if they are non-hyperbolic critical points – purely imaginary and conjugated.

In Fig. 2 the phase portrait for the ΛCDM model is presented on compactified projective plane by circle at infinity. Of course it is structurally stable. Therefore, following the Peixoto theorem, it is generic in the ensemble $\mathcal{M}$ of all dark energy models with 2-dimensional phase space because such systems form open and dense subsets.

It is also interesting that the phase space of dark energy model can be reconstructed from SN Ia data set and is topologically equivalent to the ΛCDM model. Fig. 3 represents the potential function

$$V(a(z)) = -\frac{1}{2}(1+z)^2 \left[\frac{\mathrm{d}}{\mathrm{d}z}\frac{d_L(z)}{1+z}\right]^{-2}, \tag{10}$$

reconstructed from the relation $d_L(z)$ – the luminosity distance d_L as a function of redshift z: $1+z = a^{-1}$. Such a reconstruction is possible due to the existence of the universal formula

$$\frac{d_L(z)}{1+z} = \int \frac{\mathrm{d}z'}{H(z')}, \tag{11}$$

for the flat model.

The main aims of presented discussion is searching for generic cases of brane-world models which are generic of the ensemble if brane-world models. We argue that adequate brane models should be located in the near ε-neighborhood of the ΛCDM model which is structurally stable. Therefore the global structure of their dynamics equivalent to the ΛCDM model is required. This implies that structural stability becomes requirement which contain model parameter.

II. BRANE DARK ENERGY MODELS AS A DYNAMICAL SYSTEMS. CONCLUDING REMARKS

Different brane models which offer explanations of acceleration of the current Universe can be characterized in terms of an additional term $V_{\mathrm{brane}}(a)$ as a function of the scale factor. In Table I we complete most popular brane models. They might provide answers to a number of cosmological puzzles including the issue of dark energy or type of the cosmological singularity.

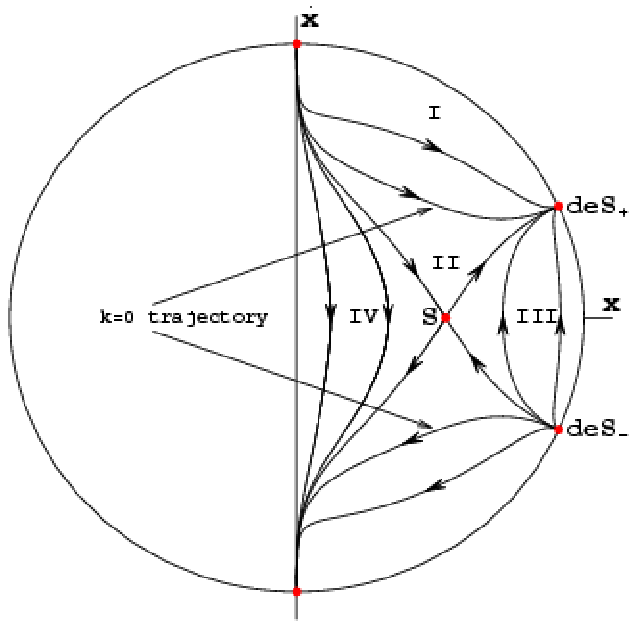


FIG. 2: The phase portrait for the ΛCDM model. In the phase portrait we have a single saddle point in the finite domain and four critical points located on the circle at infinity. They represent an initial singularity ($x = 0, \dot{x} = \infty$) or a de Sitter universe (deS). The trajectory of the flat model ($k = 0$) divides all models in to two disjoint classes: closed and open. The trajectories situated in region II confined by the upper branch of the $k = 0$ trajectory and by the separatrix going to the stable de Sitter node deS$_+$ and by the separatrix going from the initial singularity to the saddle point S correspond to the closed expanding universe. The trajectories located in regions I and II (which corresponds to the open universes) are called inflectional. Quite similarly, the trajectories situated in region III correspond to the closed universes contracting from the unstable de Sitter node towards the stable de Sitter node. The trajectories running in this region describe the closed bouncing universes. The trajectories located in region IV correspond to the oscillating closed universes expanding from the initial singularity located at ($x = 0, \dot{x} = \infty$) towards the final singularity at ($x = 0, \dot{x} = -\infty$).

The Einstein equations on the brane, in general, form a very complicated system of non-linear partial differential equations. However, the majority of most interesting models from the cosmological point of view belong to the class of homogeneous and isotropic ones, for which this complicated system of equations reduces to the form of a dynamical system. Hence in investigation of dynamics of these models the methods of dynamical systems seems to be natural. A main advantage of this methods is that we can obtain on the phase portraits global visualization of the dynamics, i.e., all evolutional paths admissible for all initial conditions. Hence we obtain asymptotic states of the system and we can analyze their type of stability (character of a critical point). The full knowledge of dynamics require analysis of dynamical behavior not only in a finite domain of the phase space but also at infinity. In our paper such an analysis was performed using the technique of Poincaré compactification of a plane by a circle at infinity. Although the application of qualitative methods in investigation of differential equations does not reveal unexpected properties of the brane models unnoticed until now, this methods offers a possibility of their classification in the terms of the evolutional paths. Moreover one detect in the phase space all global attractors and their insets. From the physical point of view it is interesting to know how large is inset or outset of limit states because the probability of an initial state of the observation to evolve asymptotically to a limit set is proportional to the volume of its inset. For example attractors have open insets, so they are probable (the inset of an attractor is called its basin).

In our analysis we mainly concentrate on the analysis of brane dynamics and its reduction to the dynamical system of a Newtonian type. But we also note that some alternative representation of the dynamics can be useful in the case of Dvali and Turner model. In some sense in this paper we extend previously introduced methods [17, 24] in large class of brane models.

Apart of detailed analysis of the brane dynamics using dynamical systems techniques we are looking for the generic dynamical systems of brane origin. Usually exceptional cases are more complicated from the mathematical point of view and they interrupt the discussion. Moreover dynamicists believe that they should not arise very often in the application, because they are exceptional ([20, p.349]). Abraham and Shaw pointed out in their famous book that a considerable portion of the history of the mathematical dynamics has been dominated by the search for generic

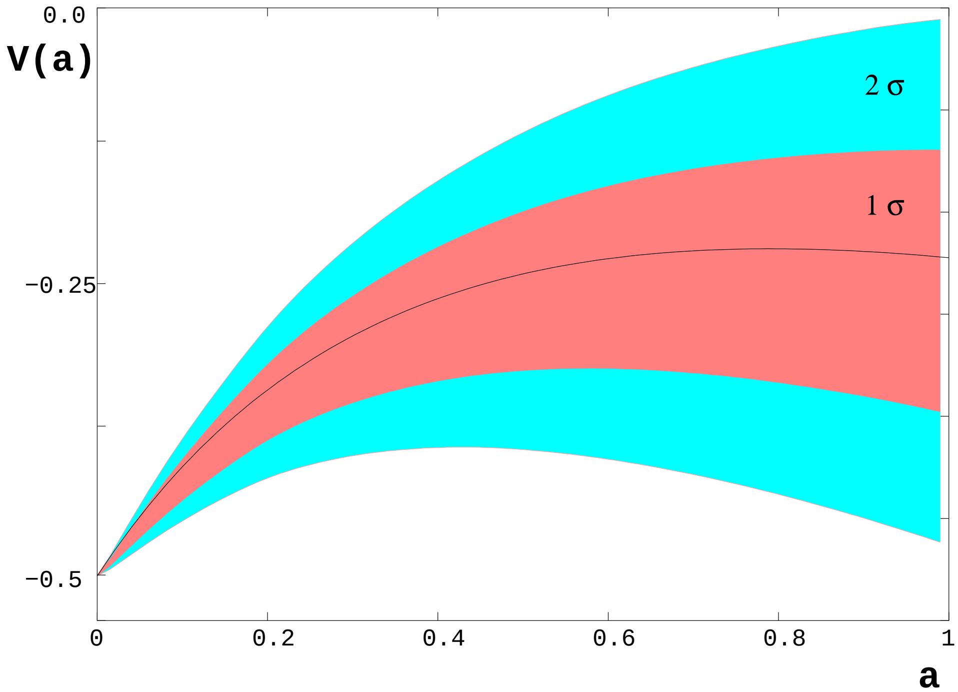


FIG. 3: The potential function as a function of the scale factor expressed in its present value $a_0 = 1$ for the reconstructed best fit model is given by the solid line. The confidence regions 1σ and 2σ are drawn around it. The phase portrait obtained from this potential (best fit) is equivalent to ΛCDM (see Fig. 2). The value of the redshift transition estimated from SNIa data (Gold sample) is about 0.38 (see [22]).

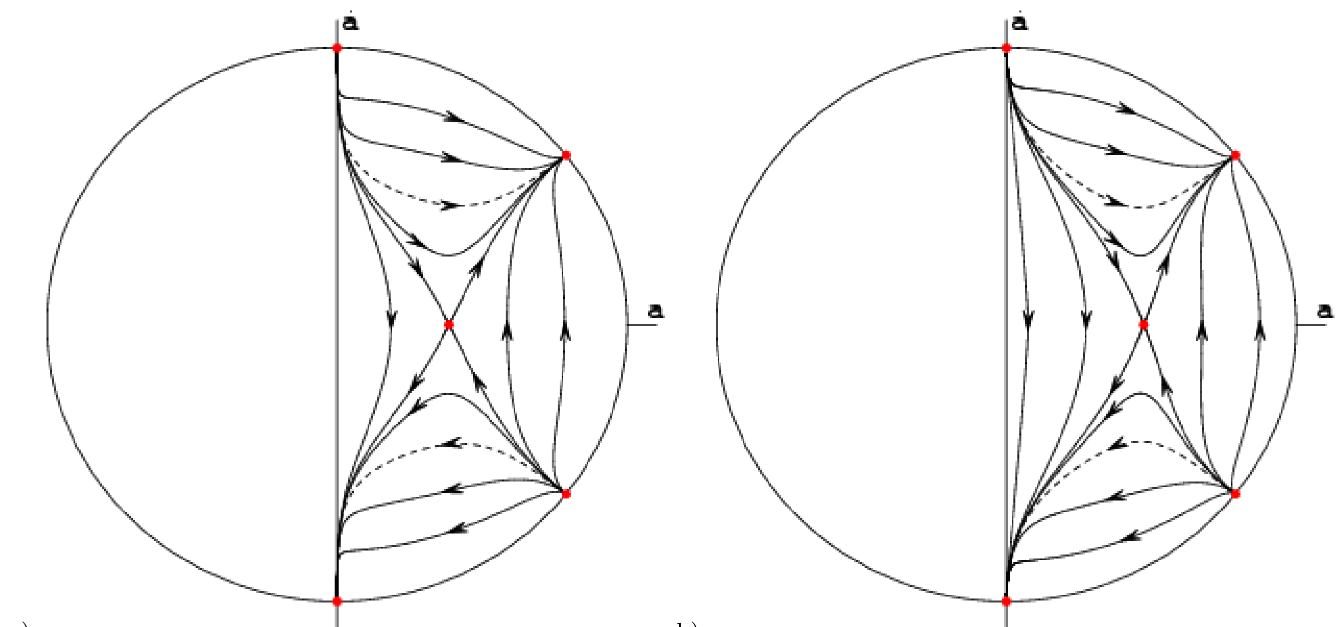


FIG. 4: Structurally stable phase portraits of Dvali-Gabadadze-Porrati model with $\Omega_{\text{m},0} = 0.3$, $\Omega_{r_c} = 0.15$ and: a) $\omega = 0$, b) $\omega = 1/3$. Dotted trajectories represent flat model trajectories $\Omega_{k,0} = 0$.

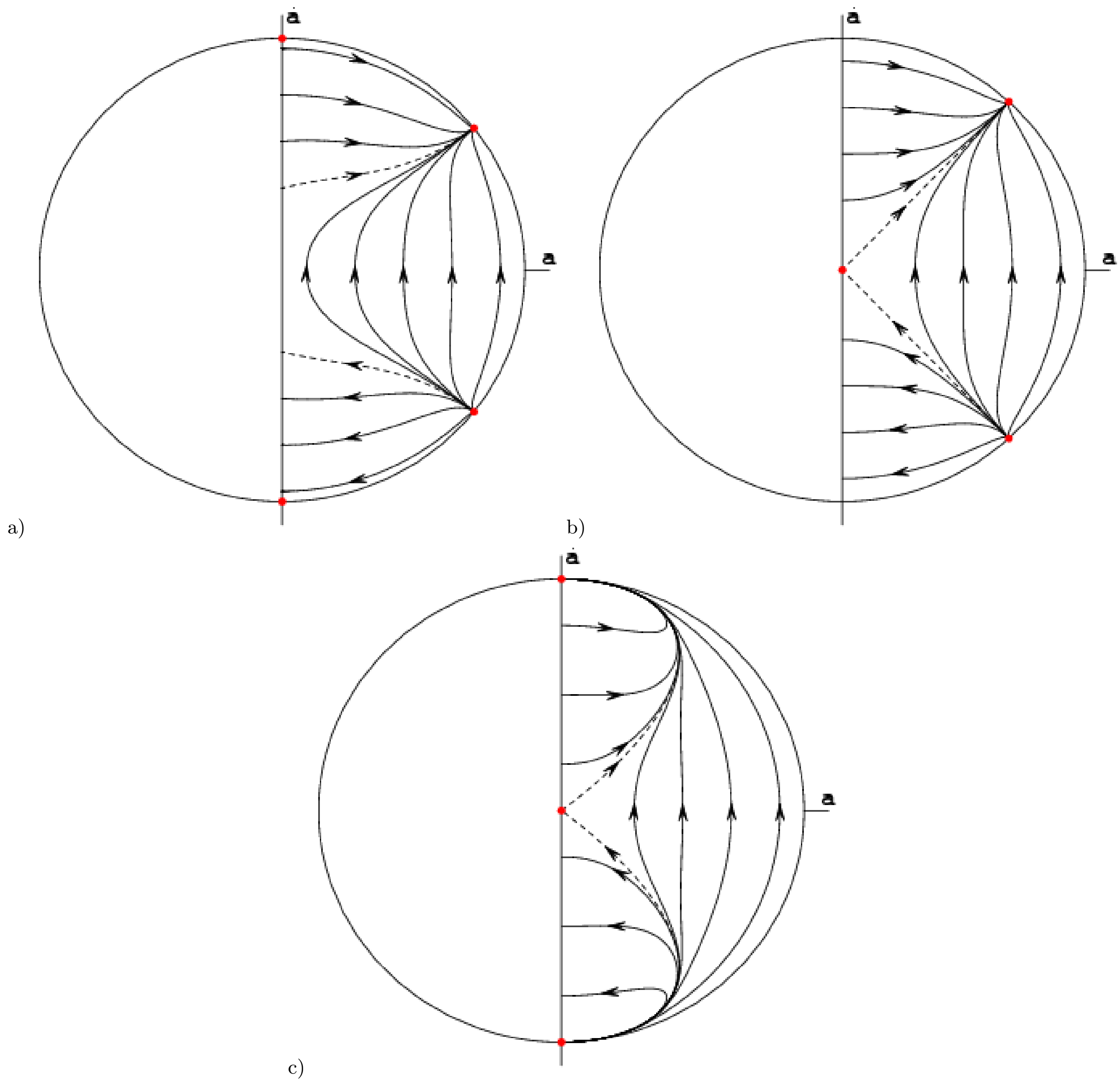


FIG. 5: Degenerated phase portraits of Dvali-Gabadadze-Porrati model with $\Omega_{\mathrm{m},0} = 0.3$, $\Omega_{r_c} = 0.15$ and: a) $\omega = -1/3$, b) $\omega = -1$ and c) $\omega = -4/3$.

properties. The idea was to characterize a class of phase portraits that are far simpler than arbitrary ones. The main motivation of the search is to complete classification of the complexity of the phase portraits (modulo exceptional cases).

This idea was achieved for dynamical systems on the plane by Peixoto due to fundamental results obtained by Russian mathematicians Andronov and Leontovich. Peixoto theorem characterize generic dynamical systems on the plane in tools of notion structural stability which was previously introduced by Andronov and Leontovich. This theorem states that structurally stable systems on the 2-dimensional closed space forms open and dense subsets in the functional space of the dynamical systems on the plane. The structurally unstable dynamical system are exceptional in this space.

The main goal of the search of structural stability of dynamical systems of brane origin on the compactified Poincaré

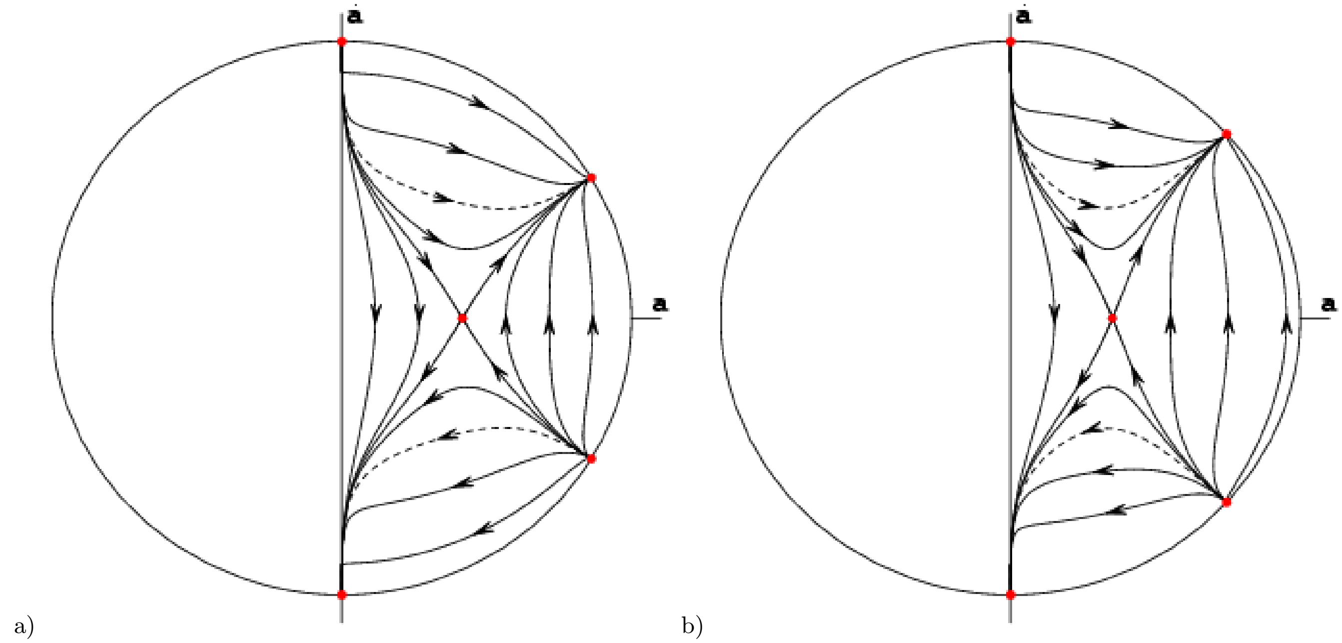


FIG. 6: Sahni-Shtanov model for $\Omega_{\mathrm{m},0} = 0.3$, $\Omega_{\Lambda,0} = 0.05$, $\Omega_{l,0} = 0.2$ and: a) brane I model (+), b) brane II model (-).

sphere is to narrow down the complexity of the portraits enough to allow classification of generic cases. This was simply achieved because for dynamical systems on the plane with polynomial right hand sides we have simple test of structural stability (instability). The notion of structural stability is intuitively very simple: A vector field has the property of structural stability if all delta perturbations of it (sufficiently small) have epsilon equivalent phase portraits. The equivalence of the phase portraits is established by homeomorphism preserving the orientation of the phase curves (topological equivalence).

It is pointed out that the brane world models have certain features that distinguish them form the other models of dark energy. They can allow for a "transient" universe acceleration phenomena which is preceded and also followed by matter domination era. Also brane models admit strange type of singularities which are known as a "quiescent" cosmological singularities. It is also possible that initial singularity can be replaced by characteristic bounce. The question is: Whether these phenomenas are attributes of a generic brane dynamical systems? Our answer is No. Typical brane models (on the plane) have global dynamics represented by phase portraits which should be topologically equivalent to the ΛCDM model.

The brane models which a topological structure of the phase space is equivalent to the ΛCDM phenomenological model provides a simple alternative to explanation of accelerating expansion of the current universe. In the concordance ΛCDM model parameter Λ is constant with respect to redshift z and matter density evolves with redshift like $(1+z)^3$. Hence appear basic problem: Why do they approximately equal each other now ? This problem called coincidence conundrum is not solved in the frame of the Λ model. Alternatively there is a possibility that there is no dark energy, but instead an infrared modification of the Friedmann equation on very large scales. To solve this problem in the frame of the DGP model it is required $r_c \sim H_0^{-1}$.

We show that the λDGP(+) model has phase portrait equivalent to the ΛCDM model which is a purely phenomenological theory offering the description of acceleration of the current universe rather than its understanding (note that they does not explain why the vacuum energy does not gravitate [25]).

The notion of structural stability allows us to formulate some general assumptions about any adequate brane models offering the explanation of acceleration of the current universe in terms of the potential function:

- the potential function has a maximum at some finite value of the scale factor or redshift; it is a redshift transition value corresponding to switching a deceleration phase to an accelerating one,
- at the late time the potential function behaves like $V \propto -a^2$ which guarantee the existence of deS$_+$ global attractor.

In our approach the dynamics is reduced to the 2-dimensional phase plane compactified by a circle at infinity. If the set of all vector fields $\boldsymbol{f} \in C^r(\mathcal{M}^2)$ $(r \geq 1)$ having a certain property contains an open dense subsets of $C^r(\mathcal{M}^2)$

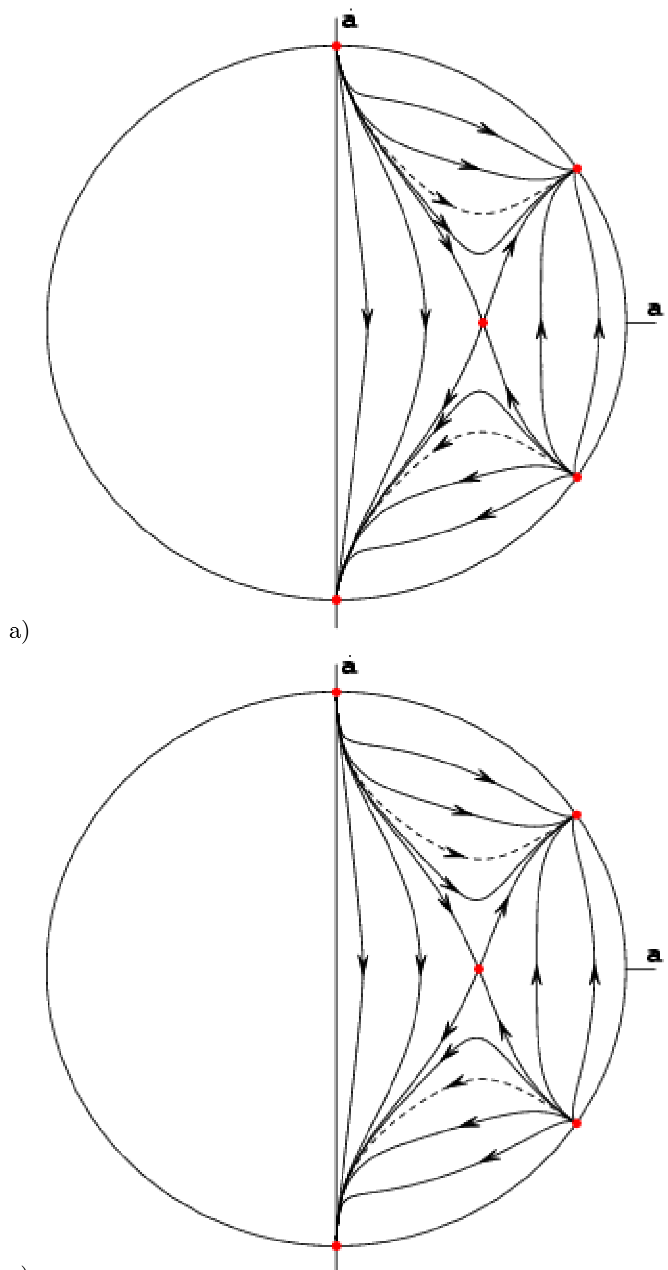


FIG. 7: Shtanov-Sahni model for $\Omega_{\mathrm{m},0} = 0.3$, $\Omega_{\Lambda,0} = 0.45$, $\Omega_{dr,0} = 0.15$ and: a) $w = 0$, $\epsilon = 1$, b) $w = 0$, $\epsilon = -1$, c) $w = -1/3$, $\epsilon = 1$, d) $w = -1/3$, $\epsilon = -1$.

then the property is called generic.

In the zoo of brane-world models all discussed previously phenomenological models can be recovered. In this way we can find counterparts of phantom models within class of Sahni-Shtanov models as well as the Cardassian one within the DGP(-) brane models.

Most interesting brane-world models should be, similar to the ΛCDM model, structurally stable. Such a postulate is satisfied by the λDGP(+) models as well as some Sahni-Shtanov models. In our opinion they should be treated seriously as candidates for explanation of current SNIa data and other concordance astronomical observations. The advantage of this models is that they offer some physical mechanism of this acceleration in contrast to the ΛCDM model.

From the physical point of view it is interesting to know whether certain subsets of ensemble contain an open and

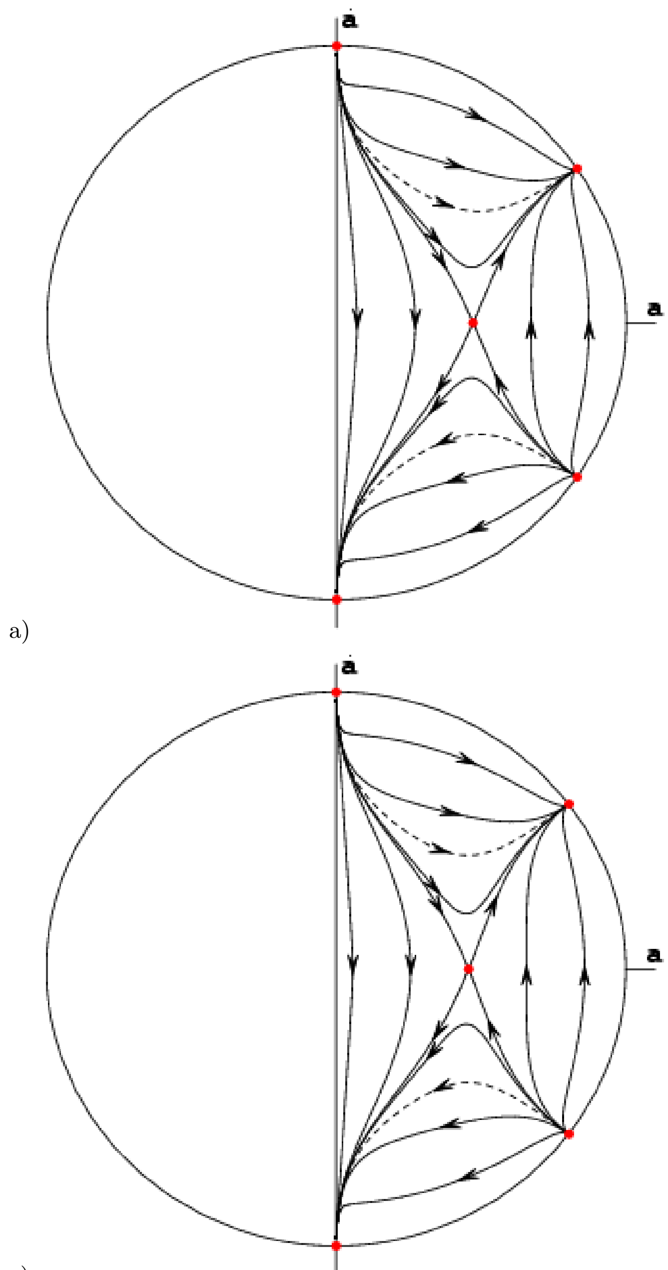


FIG. 8: Shtanov-Sahni model for $\Omega_{\mathrm{m},0} = 0.3$, $\Omega_{\Lambda,0} = 0.45$, $\Omega_{dr,0} = 0.15$ and: a) $w = -2/3$, $\epsilon = 1$, b) $w = -2/3$, $\epsilon = -1$, c) $w = -1$, $\epsilon = 1$, d) $w = -1$, $\epsilon = -1$. All models are structurally stable and therefore they are generic in the ensemble of dynamical systems on the plane of cosmological origin.

dense subsets because it means that this property is typical.

We always try to convey the features of typical garden variety of dynamical system accelerating cosmological models. The exceptional cases are usually more complicated and they in principle interrupt the discussion. This prejudice is shared by all dynamicists ([20, p.149]). We find that brane models in which instead of initial singularity there is present bounce are structurally unstable because of the center on the phase portraits. Therefore the bounce is not generic property of the evolutional scenario of accelerating brane models. Such a models require presence of double accelerating phases during the cosmological evolution. In terms of potential function of dynamical system it means that the existence of minimum before the maximum. If we consider the Sahni-Shtanov brane II model then for some set of parameters appears a quiescent singularity at which the density, pressure and the Hubble function remain finite, while all the Riemannian invariants diverge to infinity. Similar models belong to the class of non-generic cosmological

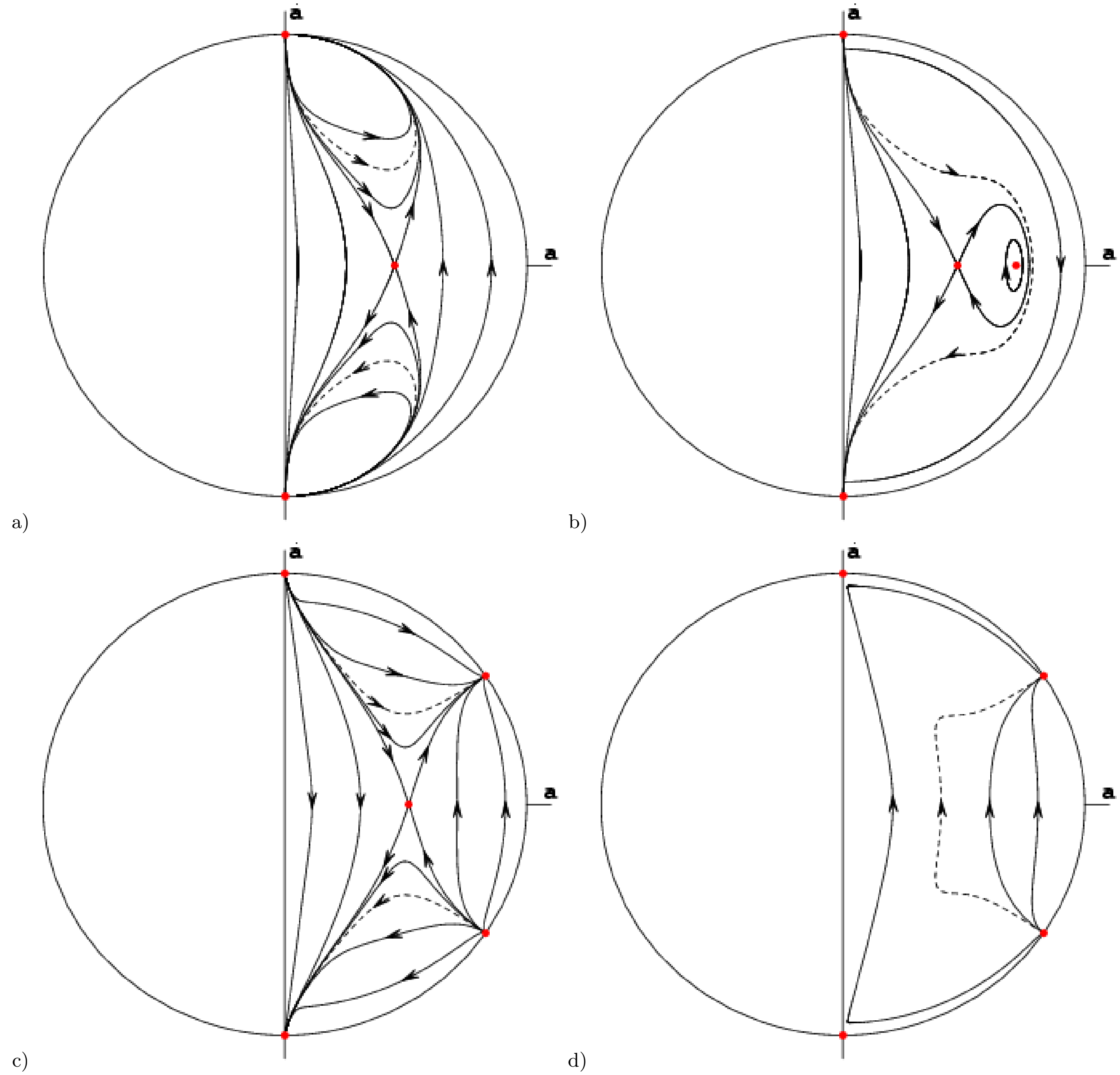
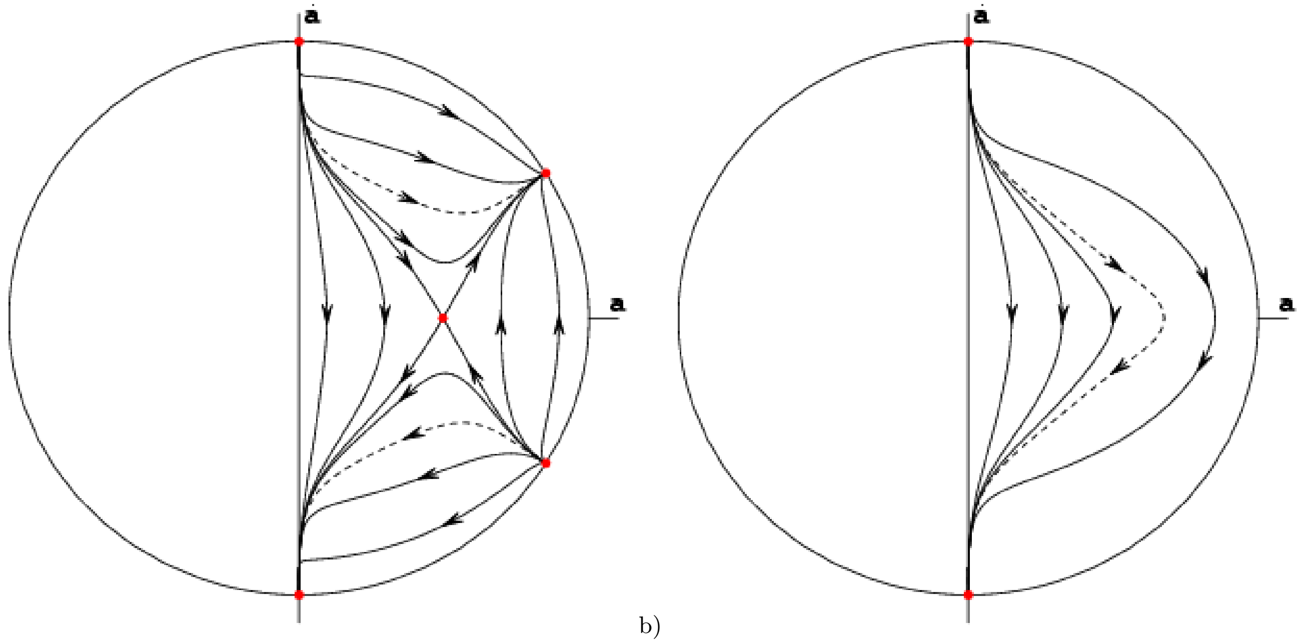


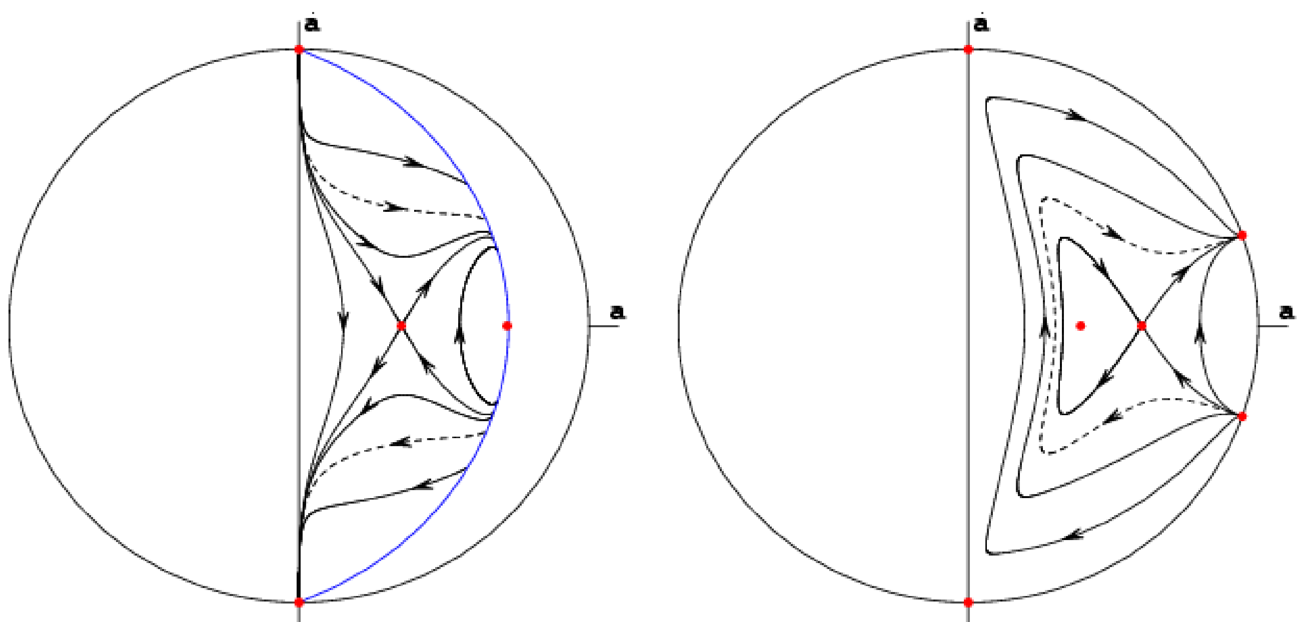
FIG. 9: Shtanov-Sahni model for $\Omega_{\mathrm{m},0} = 0.3$, $\Omega_{\Lambda,0} = 0.45$, $\Omega_{dr,0} = 0.15$ and: a) $w = -4/3$, $\epsilon = 1$, b) $w = -4/3$, $\epsilon = -1$ ($\Omega_{\Lambda,0} = 0.51$ for better separation of finite critical points), c) $w = 1/3$, $\epsilon = 1$, d) $w = 1/3$, $\epsilon = -1$.

models. Note that all models within the accelerating phase, which is only a transitional phenomenon, are non-generic too. As it was demonstrated by Sahni and Shtanov [26] there is a class of loitering brane-world models. In the phase portraits the corresponding evolutional paths are situated near the separatrices of saddle point (inflectional models). This property is generic in contrast to a quiescent future singularity.



a) b)

FIG. 10: Phase portraits for Dvali-Turner model for $\Omega_{\mathrm{m},0} = 0.45$. From Table I for $\alpha = 1$ we receive a quadratic equation for the variable $y = (H/H_0)(a/a_0)$ and hence two solutions for the potential function: a) with a plus sign, b) with a minus sign.



a) b)

FIG. 11: a) Phase portrait for Sahni–Shtanov brane I model with $\Omega_{\mathrm{m},0} = 0.2$, $\Omega_{\Lambda,0} = 0$ and $\Omega_{l,0} = 0.32$. Dashed lines denote flat model trajectories $\Omega_{k,0} = 0$. Solid line (blue in el. version) denotes maximal allowed value of a for which square root in case IV (see Table I) is positive [23] . b) Phase portrait for Shtanov–Sahni model with $\Omega_{\mathrm{m},0} = 0.3$, $\Omega_{dr,0} = 0.47$, $\Omega_{\Lambda,0} = 0.12$ and $w = 0$, $\epsilon = -1$. This class of brane–world models allows for a transient acceleration of the universe which is proceeded and followed by a matter domination era. Such a models admit quiescent singularities. Model with time-like extra dimension can also avoid cosmological singularity by a bounce.

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